MATH 127 - Final Exam - Review Sheet

Fall 2021 - Chapters 13, 14, 15, 16, and Sections 17.1-17.3

Final Exam: Monday 12/13, 4:30-7 PM

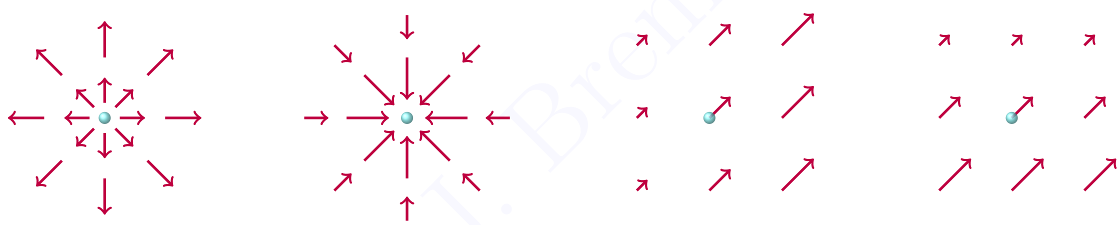
The following is a list of important concepts from the sections which were **not** covered by Midterm Exam 1 or 2. This is not a complete list of the material that you should know for the course, but it is a good indication of what will be emphasized on the exam. A thorough understanding of all of the following concepts will help you perform well on the exam. Some places to find problems on these topics are the following: in the book, in the slides, in the homework, on quizzes, and WebAssign.

Vector Fields and Line Integrals: (Sections 16.1, 16.2, 16.3, 17.1)

A **vector field** in $\mathbb{R}^n$, denoted $\vec{\mathbf{F}}$, is a function that assigns to each point $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ a vector $\vec{\mathbf{F}}(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$. The vector field $\vec{\mathbf{F}}$ is **smooth** if each of its components is continuously differentiable.

The **divergence** of a vector field $\vec{\mathbf{F}} = \langle F_1, F_2, F_3 \rangle$ is defined as

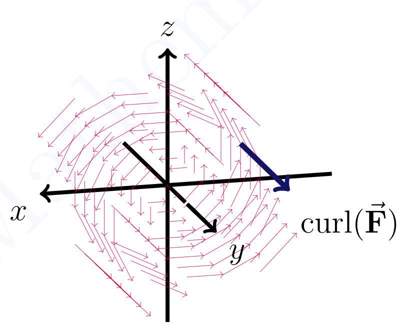
$$\operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

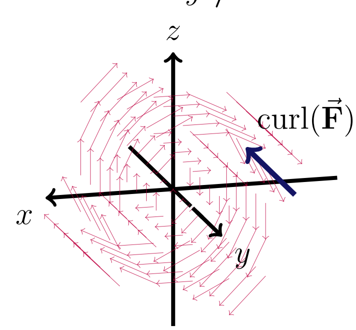


$\operatorname{div}(\vec{\mathbf{F}}) > 0$ $\quad$ $\operatorname{div}(\vec{\mathbf{F}}) < 0$ $\quad$ $\operatorname{div}(\vec{\mathbf{F}}) > 0$ $\quad$ $\operatorname{div}(\vec{\mathbf{F}}) < 0$

The **curl** of a vector field $\vec{\mathbf{F}} = \langle F_1, F_2, F_3 \rangle$ is defined as

$$\operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$



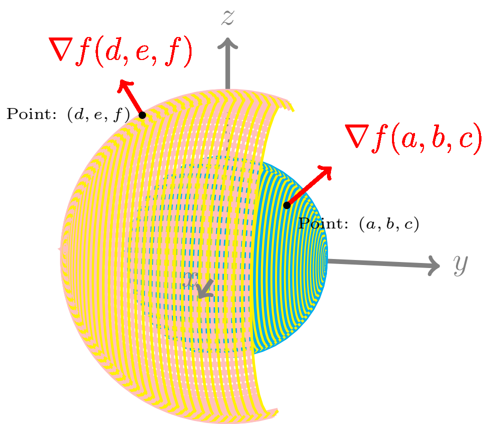


Given a differentiable function $f(x, y, z)$, its **gradient** is the vector field

$$\vec{\mathbf{F}} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

.

$\nabla f(a, b, c)$ is orthogonal to the level surface of $f(x, y, z)$ at (a, b, c).



A vector field that is the gradient of some scalar function f is called a **conservative vector field**, and the function f is called a **scalar potential function** for $\vec{\mathbf{F}}$.

If the vector field $\vec{\mathbf{F}} = \langle F_1, F_2 \rangle$ is conservative then $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$.

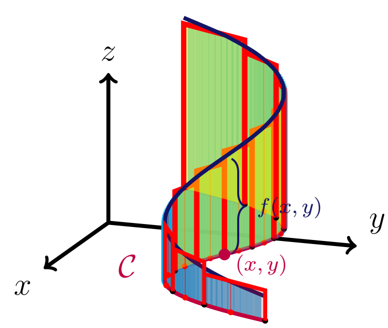
If the vector field $\vec{\mathbf{F}} = \langle F_1, F_2, F_3 \rangle$ is conservative then curl $\left(\vec{\mathbf{F}}\right) = \vec{\mathbf{0}}$ and

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

Finding the scalar potential: If curl$(\vec{\mathbf{F}}) = \vec{0}$ over a simply-connected domain, then $\vec{\mathbf{F}}$ is conservative. To find the scalar potential, compare $\displaystyle\int F_1\,dx$, $\displaystyle\int F_2\,dy$ and $\displaystyle\int F_3\,dz$. Use common terms only once. Make sure that if a term is not explicit in any of the integrals, they are consider a constant with respect to that variable.

Scalar Line Integrals: The net area under the surface $z = f(x,y)$ above the curve $\mathcal{C}$ in the xy-plane is the line integral $\displaystyle\int_{\mathcal{C}} f(x,y)\,ds$. If $\mathcal{C}$ is parametrized by $\vec{\mathbf{r}}\,(t)$ for $q \le t \le b$, then

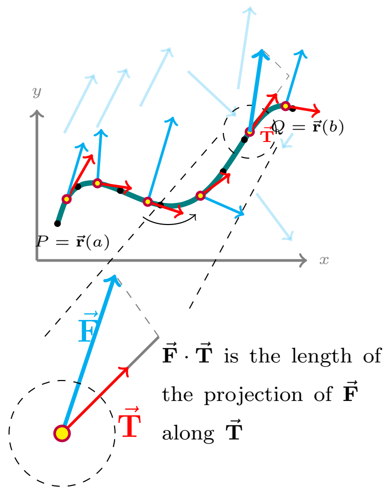
$$\int_{\mathcal{C}} f\,ds = \int_a^b f\left(\vec{\mathbf{r}}(t)\right)\|\vec{\mathbf{r}}\,'(t)\|\,dt.$$



Vector Line Integrals: The vector line integral of a vector field $\vec{\mathbf{F}}$ over an oriented curve $\mathcal{C}$ is the scalar line integral of the tangential component of $\vec{\mathbf{F}}$.

If $\vec{\mathbf{F}} = \langle P, Q, R \rangle$ and $\mathcal{C}$ is parameterized by $\vec{\mathbf{r}}(t)$, then

$$\begin{aligned}\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} &= \int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\,ds \\ &= \int_a^b \vec{\mathbf{F}}\left(\vec{\mathbf{r}}\,(t)\right) \cdot \vec{\mathbf{r}}\,'(t)\,dt \\ &= \int P\,dx + Q\,dy + R\,dz\end{aligned}$$

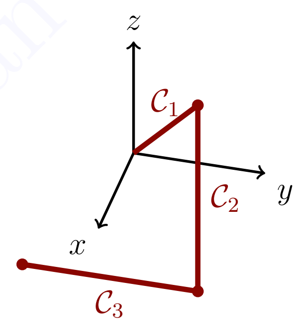


The work done in moving an object from $\vec{\mathbf{r}}\,(a)$ to $\vec{\mathbf{r}}\,(b)$ along $\mathcal{C}$ is $\displaystyle\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

$\mathcal{C}$ is **piecewise-smooth** if $\mathcal{C}$ is the union of a finite number of smooth curves $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$.

$$\int_{\mathcal{C}} f\,ds = \int_{\mathcal{C}_1} f\,ds + \int_{\mathcal{C}_2} f\,ds + \ldots + \int_{\mathcal{C}_n} f\,ds$$

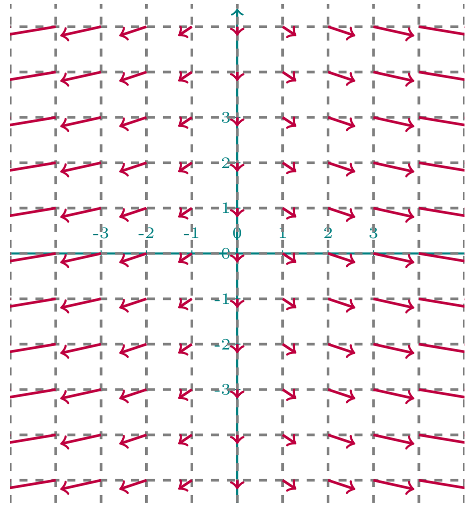
$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \ldots + \int_{\mathcal{C}_n} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$



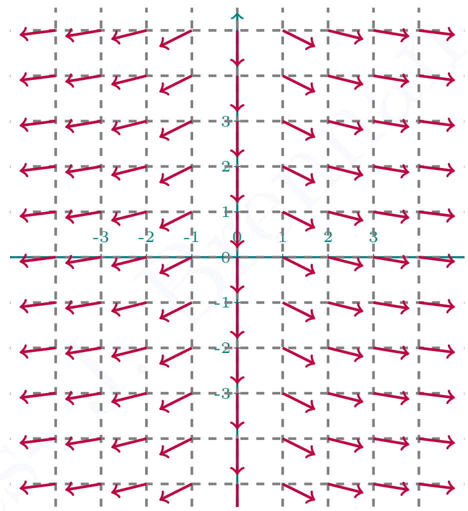
Exercises:

1. $f(x, y) = x^2 - y$ is a potential function for $\vec{F}$. Find and sketch $\vec{F}$.

$$\vec{F}(x, y) = \langle 2x, -1 \rangle$$

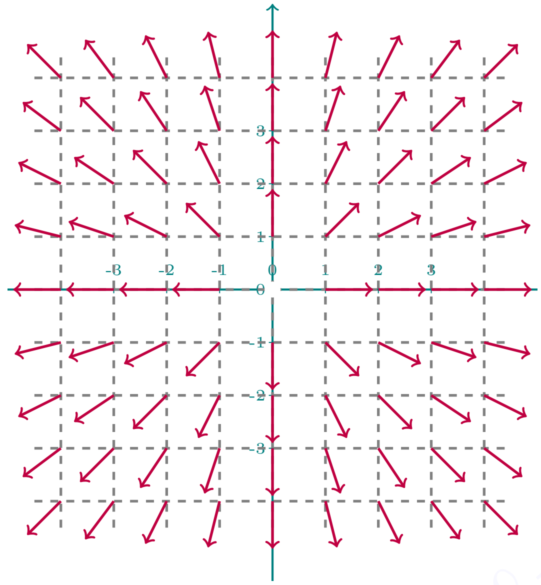


OR



2. $f(x,y) = \sqrt{x^2+y^2}$ is a potential function for $\vec{F}$. Find and sketch $\vec{F}$.

$$\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$



3. Calculate the curl and divergence of the vector fields:

(A) $\vec{F}(x,y,z) = \langle xyz, 0, -x^2y \rangle$

$$\operatorname{div}\left(\vec{F}\right) = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-x^2y) = yz$$

$$\operatorname{curl}\left(\vec{F}\right) = \left\langle \frac{\partial}{\partial y}(-x^2y) - \frac{\partial}{\partial z}(0), \frac{\partial}{\partial z}(xyz) - \frac{\partial}{\partial x}(-x^2y), \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(xyz) \right\rangle = \langle -x^2, 3xy, -xz \rangle$$

(B) $\vec{F}(x,y,z) = \langle 0, \cos(xz), -\sin(xy) \rangle$

$\operatorname{div}\left(\vec{F}\right) = 0$ $\qquad$ $\operatorname{curl}\left(\vec{F}\right) = \langle -x\cos(xy) + x\sin(xz), y\sin(xy), -z\sin(xz) \rangle$

(C) $\nabla\left(e^{xyz}\right)$

$\nabla\left(e^{xyz}\right) = \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle$ $\qquad$ $\vec{F}$ is conservative, so $\operatorname{curl}\left(\vec{F}\right) = \vec{0}$

$\operatorname{div}\left(\vec{F}\right) = \left(y^2z^2 + x^2z^2 + x^2y^2\right)e^{xyz}$

4. $f(x,y,z) = xyz$ and $\mathcal{C}$ is parametrized $\vec{r}(t) = \langle 2\sin(t), t, -2\cos(t) \rangle$ for $[0,\pi]$. Evaluate $\displaystyle\int_{\mathcal{C}} f\, ds$.

Since $|\vec{r}'(t)| = |\langle 2\cos(t), 1, 2\sin(t)\rangle| = \sqrt{5}$,

$$\int_{\mathcal{C}} f(x,y,z)\,ds = \int_0^{\pi} (2\sin(t))(t)(-2\cos(t))\sqrt{5}\,dt = -4\sqrt{5}\int_0^{\pi} t\sin(t)\cos(t)\,dt = \sqrt{5}\pi$$

5. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geq 0$. If the linear density is a constant k, find the mass of the wire.

Parametrize the wire as $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The density of the wire at (x,y) is $\rho(x,y) = k$. To find the mass, we take the scalar linear integral.

Since $|\vec{r}'(t)| = |\langle -2\sin(t), 2\cos(t)\rangle| = 2$,

$$\textbf{Mass of Wire:} \qquad \int_{\mathcal{C}} \rho(x,y)\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2k\,dt = 2k\pi.$$

6. Evaluate the vector line integral of $\vec{F}(x,y,z) = \langle x+y, y-z, z^2\rangle$ over $\vec{r}(t) = \langle t^2, t^3, t^2\rangle$ on $[0,1]$.

$$\begin{aligned}\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} &= \int_0^1 \left\langle t^2+t^3, t^3-t^2, t^4\right\rangle \cdot \left\langle 2t, 3t^2, 2t\right\rangle\,dt \\ &= \int_0^1 2t(t^2+t^3) + 3t^2(t^3-t^2) + 2t(t^4)\,dy \\ &= \int_0^1 5t^5 - t^4 + 2t^3\,dt = \frac{17}{15}\end{aligned}$$

7. Calculate the work done by the vector field $\vec{F}(x,y,z) = \langle -y\sin(z), x\sin(z), xy\cos(z)\rangle$ in moving a particle around the circle cut from $x^2+y^2+z^2 = 9$ by $z = -1$, clockwise as viewed from above.

The curve cut from the sphere by the plane satisfies the equations $x^2 + y^2 = 8$, $z = -1$. Parametrized with clockwise orientation as

$$\vec{r}(t) = \left\langle \sqrt{8}\sin(t), \sqrt{8}\cos(t), -1\right\rangle \qquad \vec{r}'(t) = \left\langle \sqrt{8}\cos(t), -\sqrt{8}\sin(t), 0\right\rangle \qquad t \in [0, 2\pi]$$

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

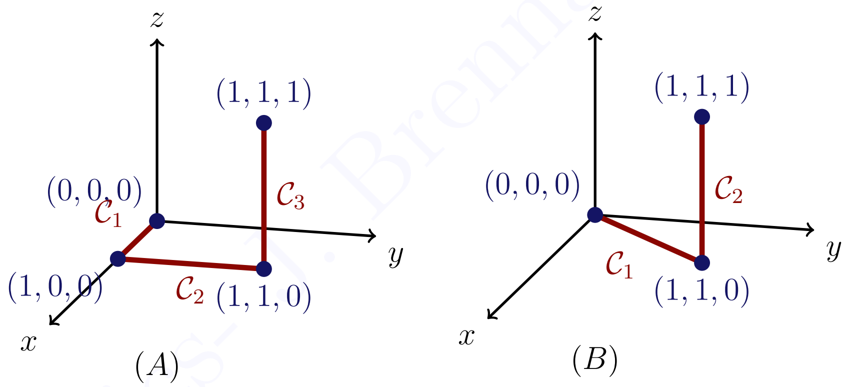
$$= \int_0^{2\pi} \left\langle \sqrt{8}\cos(t)\sin(1), -\sqrt{8}\sin(t)\sin(1), 8\cos(t)\sin(t)\cos(1) \right\rangle \cdot \left\langle \sqrt{8}\cos(t), -\sqrt{8}\sin(t), 0 \right\rangle dt$$

$$= \int_0^{2\pi} 8\sin(1)\, dt = 16\pi \sin(1)$$

8. Integrate $f(x, y, z) = x^2 + y - z$ over the following paths:

   (A) The path consisting of line segments from $(0, 0, 0)$ to $(1, 0, 0)$, then to $(1, 1, 0)$, and finally to $(1, 1, 1)$.

   (B) The path consisting of line segments from $(0, 0, 0)$ to $(1, 1, 0)$ and then to $(1, 1, 1)$.



(A) The curve consists of three paths:

$$\vec{r}_1(t) = t\langle 1, 0, 0\rangle + (1 - t)\langle 0, 0, 0\rangle = \langle t, 0, 0\rangle \qquad |\vec{r}_1\,'| = 1 \qquad t \in [0, 1]$$

$$\vec{r}_2(t) = t\langle 1, 1, 0\rangle + (1 - t)\langle 1, 0, 0\rangle = \langle 1, t, 0\rangle \qquad |\vec{r}_2\,'| = 1 \qquad t \in [0, 1]$$

$$\vec{r}_3(t) = t\langle 1, 1, 1\rangle + (1 - t)\langle 1, 1, 0\rangle = \langle 1, 1, t\rangle \qquad |\vec{r}_3\,'| = 1 \qquad t \in [0, 1]$$

$$\int_{\mathcal{C}} f\, ds = \int_0^1 f(\vec{r}_1)\, |\vec{r}_1\,'|\, dt + \int_0^1 f(\vec{r}_2)\, |\vec{r}_2\,'|\, dt + \int_0^1 f(\vec{r}_3)\, |\vec{r}_3\,'|\, dt$$

$$= \int_0^1 t^2\, dt + \int_0^1 t + 1\, dt + \int_0^1 2 - t\, dt = 10/3$$

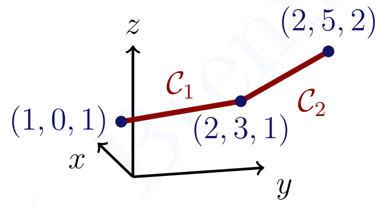
(B) The curve consists of two paths:

$$\vec{r}_1(t) = t\langle 1,1,0\rangle + (1-t)\langle 0,0,0\rangle = \langle t,t,0\rangle \qquad |\vec{r}_1\,'| = \sqrt{2} \qquad t\in[0,1]$$

$$\vec{r}_2(t) = t\langle 1,1,1\rangle + (1-t)\langle 1,1,0\rangle = \langle 1,1,t\rangle \qquad |\vec{r}_2\,'| = 1 \qquad t\in[0,1]$$

$$\begin{aligned}\int_{\mathcal{C}} f\,ds &= \int_0^1 f(\vec{r}_1)\,|\vec{r}_1\,'|\,dt + \int_0^1 f(\vec{r}_2)\,|\vec{r}_2\,'|\,dt \\ &= \int_0^1 (t^2+t)\sqrt{2}\,dt + \int_0^1 2-t\,dt = \frac{5\sqrt{2}+9}{6}\end{aligned}$$

9. Evaluate $\displaystyle\int_{\mathcal{C}} (x+yz)\,dx + 2x\,dy + (xyz)\,dz$ where $\mathcal{C}$ consists of the line segments $\mathcal{C}_1$: from $(1,0,1)$ to $(2,3,1)$ and $\mathcal{C}_2$: from $(2,3,1)$ to $(2,5,2)$.



$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ where $\mathcal{C}_1$ is parametrized by $\vec{r}_1$ and $\mathcal{C}_2$ by $\vec{r}_2$.

$$\vec{r}_1(t) = (1-t)\langle 1,0,1\rangle + t\langle 2,3,1\rangle = \langle t+1, 3t, 1\rangle \qquad t\in[0,1] \qquad \vec{r}\,'(t) = \langle 1,3,0\rangle$$

$$\vec{r}_2(t) = (1-t)\langle 2,3,1\rangle + t\langle 2,5,2\rangle = \langle 2, 2t+3, t+1\rangle \qquad t\in[0,1] \qquad \vec{r}\,'(t) = \langle 0,2,1\rangle$$

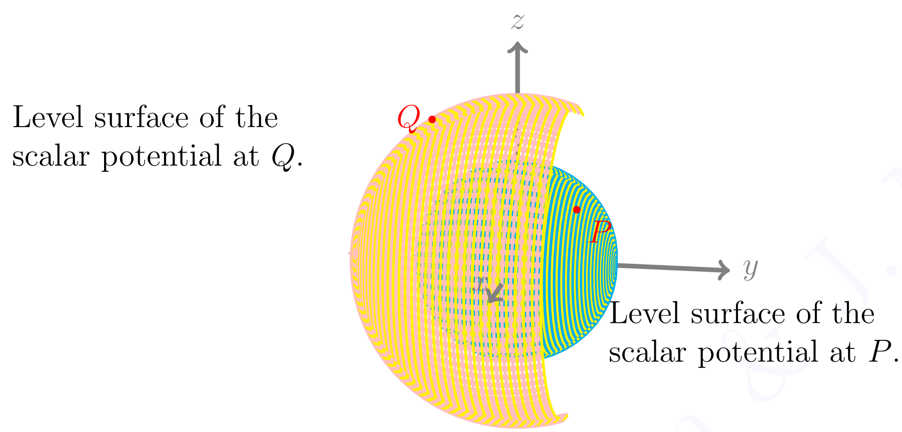
The integral is a vector line integral of $\vec{F}(x,y,z) = \langle x+yz, 2x, xyz\rangle$ over $\mathcal{C}$.

$$\begin{aligned}\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} &= \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r} + \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{r} \\ &= \int_0^1 \left\langle 4t+1, 2t+2, 3t^2+3t\right\rangle\cdot\langle 1,3,0\rangle\,dt + \int_0^1 \left\langle 2t^2+5t+5, 4, 4t^2+10t+6\right\rangle\cdot\langle 0,2,1\rangle\,dt \\ &= \int_0^1 10t+7\,dt + \int_0^1 4t^2+10t+14\,dt = \frac{97}{3}\end{aligned}$$

Fundamental Theorems of Line Integrals: (Sections 16.3, 17.1)

Fundamental Theorem for Conservative Vector Fields: Assume that $\vec{\mathbf{F}} = \nabla f$ on a domain $\mathcal{D}$. For *any* curve $\mathcal{C}$ from P to Q in $\mathcal{D}$,

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = f(Q) - f(P).$$

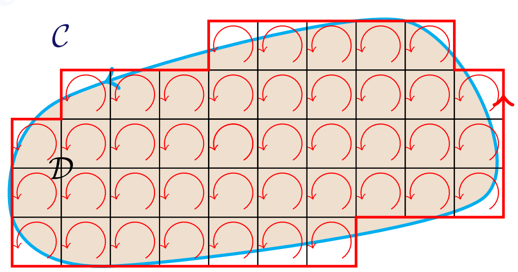


A curve is *simple* if it does not intersect itself. It is *closed* if it begins and ends at the same point. A parameterization of a simple, closed curve is *positively oriented* if the point moves counterclockwise.

Green's Theorem: If $\mathcal{D}$ is a domain whose boundary $\partial\mathcal{D}$ is a simple, closed curve with positive orientation, then

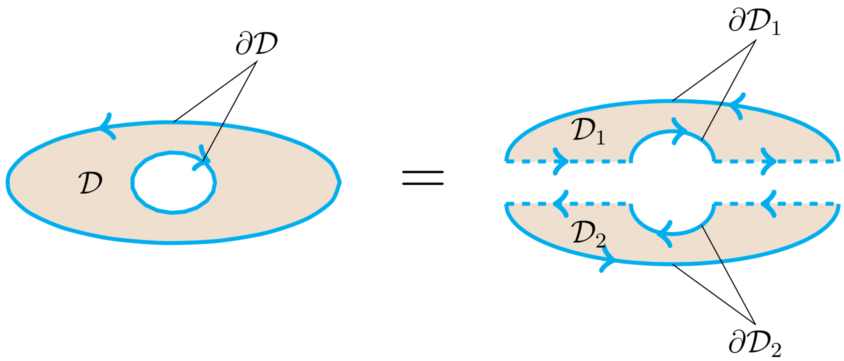
$$\int_{\partial\mathcal{D}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\partial\mathcal{D}} P\,dx + Q\,dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \iint_{\mathcal{D}} \operatorname{curl}(\vec{\mathbf{F}})_z\,dA$$

The *boundary* of a surface $\mathcal{S}$ is denoted $\partial\mathcal{S}$. When $\mathcal{S}$ is oriented, the induced boundary orientation is the direction which keeps the surface on the left if you were to walk along the boundary with your feet on the curve and your head pointed in the direction of the orientation of the surface.



Regions With Holes: For a connected region with holes, the boundary consists of two or more closed curves. Every part of the boundary must be oriented to keep the region on the **left**.

Outside boundary: counterclockwise. Inside boundary: clockwise.



Exercises:

1. Find the work done by $\nabla\left((x+y)^2\right)$ counterclockwise around the circle $x^2+y^2=4$ from $(2,0)$ to $(0,-2)$.

Parametrize the curve by $\vec{r}(t)=\langle 2\cos(t), 2\sin(t)\rangle$ on $[0, 3\pi/2]$. $\vec{r}\,'(t)=\langle -2\sin(t), 2\cos(t)\rangle$.

$\vec{F}(x,y)=\nabla\left((x+y)^2\right)=\langle 2x+2y, 2x+2y\rangle$.

$$\begin{aligned}\int_{\mathcal{C}}\vec{F}\cdot d\vec{r} &= \int_0^{3\pi/2}\langle 2\cos(t)+2\sin(t), 2\cos(t)+2\sin(t)\rangle\cdot\langle -2\sin(t), 2\cos(t)\rangle\, dt\\ &= \int_0^{3\pi/2} -4\sin^2(t)+4\cos^2(t)\, dt = 0\end{aligned}$$

Alternatively:

Notice $\vec{\mathbf{F}}(x,y)=\nabla((x+y)^2)$ is conservative and its scalar potential is $f(x,y)=(x+y)^2$. Now by fundamental theorem of conservative vector integrals,

$$\int_{\mathcal{C}}\nabla((x+y)^2)\cdot d\vec{\mathbf{r}} = f(2,0)-f(0,-2)=0$$

2. Consider the vector field $\vec{F}(x,y,z)=\langle\cos(z), -1, -x\sin(z)\rangle$.

   (A) Is $\vec{F}$ is **conservative** on $\mathbb{R}^3$?

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \cos(z) & -1 & -x\sin(z)\end{vmatrix}=\langle 0,0,0\rangle \text{ and the domain of } \vec{F} \text{ is all } \mathbb{R}^3.$$

   (B) Find the scalar **potential** function f for the gradient field $\vec{F}$.

- $f(x, y, z) = \displaystyle\int \cos(z)\, dx = x\cos(z) + C_1(y, z).$
- $f(x, y, z) = \displaystyle\int -1\, dy = -y + C_2(x, z).$
- $f(x, y, z) = \displaystyle\int -x\sin(z)\, dz = x\cos(z) + C_3(x, y).$

$f(x, y, z) = x\cos(z) - y + C$

(C) Evaluate $\displaystyle\int_C \vec{F} \cdot\, d\vec{r}$ along C given by $\vec{r}(t) = \langle e^t, e^{2t}, t\rangle$ from point $(1, 1, 0)$ to $(e^{\pi}, e^{2\pi}, \pi)$.

$$\int_C \vec{F} \cdot d\vec{r} = f(e^{\pi}, e^{2\pi}, \pi) - f(1, 1, 0) = e^{\pi}\cos(\pi) - e^{2\pi} - \Big((1)\cos(0) - 1\Big) = -e^{\pi} - e^{2\pi}$$

3. Calculate the work done by the vector field $\vec{F}(x, y, z) = \left\langle 1, -\sqrt{z/y}, -\sqrt{y/z}\right\rangle$ in moving a particle from $(1, 1, 1)$ to $(10, 3, 3)$ along a path which stays within the first octant.

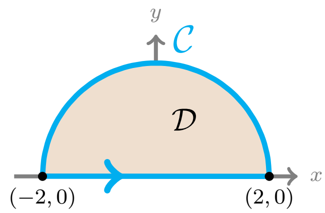
$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & -\sqrt{z/y} & -\sqrt{y/z} \end{vmatrix} = \vec{0}$$

Since $\operatorname{curl}\Big(\vec{F}\Big) = \vec{0}$ on $\mathbb{R}^3 - \{(0, y, 0)\,|\, y \in \mathbb{R}^3\} - \{(0, 0, z)\,|\, z \in \mathbb{R}^3\}$, the first octant is entirely in the domain of $\vec{\mathbf{F}}$. Therefore, $\vec{\mathbf{F}}$ is conservative within the first octant.

The function $f(x, y, z) = x - 2\sqrt{yz}$ is a potential function for $\vec{F}$. Using the Fundamental Theorem of Conservative Vector Fields,

$$\int_{(1,1,1)}^{(10,3,3)} \vec{F} \cdot d\vec{r} = f(10, 3, 3) - f(1, 1, 1) = 5$$

4. A particle starts at the point $(-2,0)$, moves along the x-axis to $(2,0)$, and the along the semicircle to $y=\sqrt{4-x^2}$ to the starting point. How much work was done by the vector field $\vec{F}(x,y) = \langle x, x^3+3xy^2\rangle$ in moving the particle along the path.

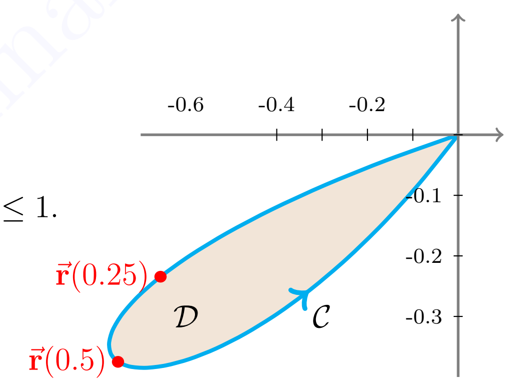


The curve moves counterclockwise about the semicircle. Using Green's Theorem,

$$\begin{aligned}\oint_{\partial\mathcal{D}} \vec{F}\cdot d\vec{r} &= \iint_{\mathcal{D}} \operatorname{curl}\left(\vec{F}\right)_z \, dA \\ &= \iint_{\mathcal{D}} \frac{\partial}{\partial x}(x^3+3xy^2) - \frac{\partial}{\partial y}(x)\, dA \\ &= \iint_{\mathcal{D}} 3(x^2+y^2)\, dA \\ &= 12\pi\end{aligned}$$

5. Let $\mathcal{C}$ be the simply closed curve defined by

$$\mathcal{C} : \vec{r}(t) = \langle -2t(t-1)(t-2),\ t(t-1)(t+1)\rangle \ \text{ for } 0 \le t \le 1.$$



(A) Find $\mathbf{\vec{r}}(0)$, $\mathbf{\vec{r}}(0.25)$ and $\mathbf{\vec{r}}(0.5)$. Then confirm that the parameterization transverses the curve in counterclockwise orientation.

$\mathbf{\vec{r}}(0)$	$(0,0)$
$\mathbf{\vec{r}}(0.25)$	$\simeq(-0.65625,-0.23438)$
$\mathbf{\vec{r}}(0.5)$	$\simeq(-0.75,-0.375)$

As t grows, the points on the curve move clockwise.

(B) For $\vec{F}(x,y)=\langle 0, x\rangle$, compute $\oint_{\partial D} \vec{F}\cdot d\vec{r}$.

$$\begin{aligned}= \oint_{\mathcal{C}} x\, dy &= \int_0^1 x\,\frac{dy}{dt}\, dt \\ &= \int_0^1 -2(t^3-3t^2+2t)(3t^2-1)\, dt \\ &= \int_0^1 -6t^5+18t^4-10t^3-6t^2+4t\, dt = \frac{1}{10}.\end{aligned}$$

The value computed is clockwise so $\oint_{\partial D} \vec{F} \cdot d\vec{r} = \frac{1}{10}$.

(C) For $\vec{F}(x, y) = \langle 0, x\rangle$, find $\text{curl}(\vec{\mathbf{F}})$; use the value to simplify $\iint_{\mathcal{D}} \text{curl}_z(\vec{\mathbf{F}})\, dA$.

$\text{curl}(\vec{\mathbf{F}}) = \vec{k}$ so $\iint_{\mathcal{D}} \text{curl}_z(\vec{\mathbf{F}})\, dA = \iint_{\mathcal{D}} 1\, dA = \text{area } (\mathcal{D})$

(D) Use Green's Theorem to find the area entrapped in the simple closed curve $\mathcal{C}$.

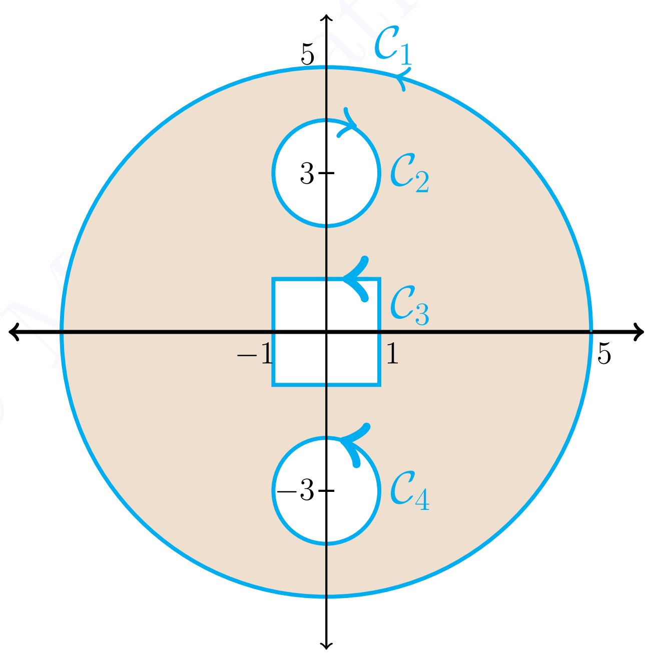
According to part (B) and (C), the area is $\frac{1}{10}$

Note: The region $\mathcal{D}$ is simple, but it is difficult to impossible to express it in a form suitable for an iterated integral. Instead, use Green's theorem and the fact that $\text{curl}_z\left(\vec{F}\right) = 1$. To find the area, we need to compute

$$\iint_{\mathcal{D}} \text{curl}_z\left(\vec{F}\right)\, dA = \oint_{\mathcal{C}} \vec{F} \cdot d\vec{r}$$

6. Find $\int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r}$ if $\text{curl}\left(\vec{F}\right)_z = 6$ in the region defined by the 4 curves and

$$\int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} = 3 \qquad \int_{\mathcal{C}_3} \vec{F} \cdot d\vec{r} = 7 \qquad \int_{\mathcal{C}_4} \vec{F} \cdot d\vec{r} = \pi$$



The boundary of $\mathcal{D}$ has a positive orientation represented as $\mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_3 - \mathcal{C}_4$. By Green's Theorem,

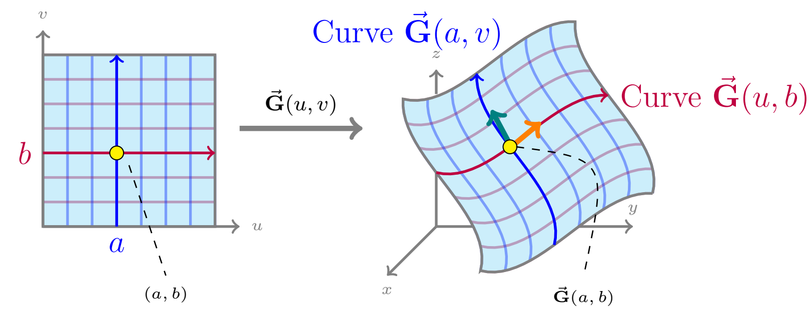
$$\oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} + \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} - \int_{\mathcal{C}_3} \vec{F} \cdot d\vec{r} - \int_{\mathcal{C}_4} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{D}} \operatorname{curl}\left(\vec{F}\right)_z \, dA$$

Therefore,

$$\oint_{\mathcal{C}_1} \vec{F} \cdot \, d\vec{r} = \iint_{\mathcal{D}} 6 \, dA - 3 + 7 + \pi = 6(\text{Area}(\mathcal{D})) + (4 + \pi) = 4 + \pi + 6(23\pi - 4) = 139\pi - 20$$

Surface Integrals: (Sections 16.4 and 16.5)

A curve is *smooth* if it has a parametrization $\vec{\mathbf{r}}(t)$ where $\vec{\mathbf{r}}\,'$ is continuous. A parametrization is *regular* if $\vec{\mathbf{r}}\,'$ is nonzero.



A surface is *smooth* if it has a parametrization $\vec{\mathbf{G}}(u, v)$ where $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ is continuous. A parametrization is *regular* if $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ is nonzero.

Let f be a scalar function and $\vec{\mathbf{F}}$ a vector field.

Scalar Line Integral along a smooth curve $\mathcal{C}$ with a regular parametrization $\vec{\mathbf{r}}\,(t)$ on $[a, b]$.

$$\int_{\mathcal{C}} f\,ds = \int_a^b f(\vec{\mathbf{r}}\,(t))\,\|\vec{\mathbf{r}}\,'(t)\|\,dt$$

If $f = 1$ then $\displaystyle\int_{\mathcal{C}} f\,ds$ is the arclength of $\mathcal{C}$.

Vector Line Integral, or work done by a vector field, along an oriented curve $\mathcal{C}$:

$$\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}\,(\vec{\mathbf{r}}\,(t)) \cdot \vec{\mathbf{r}}\,'(t)\,dt$$

Scalar Surface Integral over a smooth surface $\mathcal{S}$ with a regular parametrization $\vec{\mathbf{G}}(u, v)$ on $\mathcal{R}$:

$$\iint_{\mathcal{S}} f\,dS = \iint_{\mathcal{R}} f(\vec{\mathbf{G}}(u, v))\,\|\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\|\,dA$$

If $f = 1$ then $\displaystyle\iint_{\mathcal{S}} f\,dS$ is the surface area of $\mathcal{S}$.

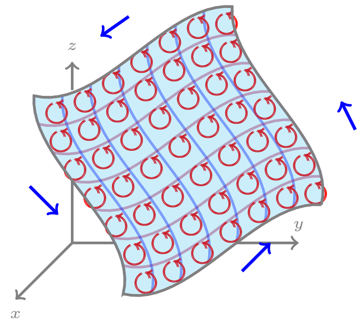
Vector Surface Integral or **flux** of a vector field $\vec{\mathbf{F}}$ through an oriented surface $\mathcal{S}$:

$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{R}} \vec{\mathbf{F}}\left(\vec{\mathbf{G}}(u, v)\right) \cdot \left(\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v\right)\,dA$$

Fundamental Theorems of Surface Integrals: (Sections 17.2-17.3)

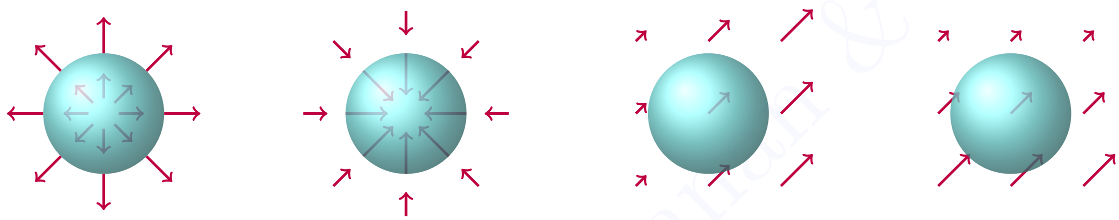
Stokes' Theorem: Let $\mathcal{S}$ be an oriented surface with smooth, simple closed boundary curves. Let $\vec{\mathbf{F}}$ be a vector field whose components have continuous partial derivatives.

$$\int_{\partial\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_{\mathcal{S}} \operatorname{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$$



The Divergence Theorem: Let $\mathcal{S}$ be a closed surface that encloses a solid $\mathcal{T}$ in $\mathbb{R}^3$. Assume that $\mathcal{T}$ is a piecewise smooth and is oriented by normal vectors pointing to the outside of $\mathcal{T}$. Let $\vec{\mathbf{F}}$ be a vector field whose domain contains $\mathcal{T}$.

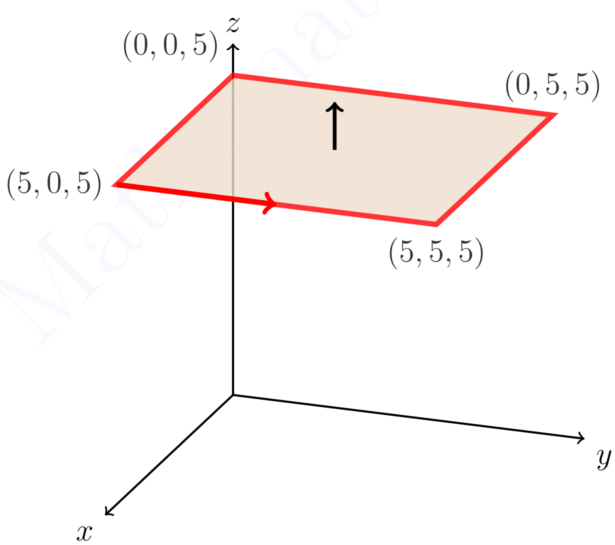
$$\iint_{\mathcal{S}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_{\mathcal{T}} \operatorname{div}\left(\vec{\mathbf{F}}\right)\, dV$$



$\operatorname{div}(\vec{\mathbf{F}}) > 0$ $\qquad$ $\operatorname{div}(\vec{\mathbf{F}}) < 0$ $\qquad$ $\operatorname{div}(\vec{\mathbf{F}}) > 0$ $\qquad$ $\operatorname{div}(\vec{\mathbf{F}}) < 0$

Exercises:

1. Compute the flux of $\vec{F}(x, y, z) = \langle 3, 4, 5\rangle$ through each of the rectangular regions below, assuming each is oriented as shown. Draw an arrow on the boundary of each surface, $\mathcal{D}$, so that the flux computed is equal to $\oint_{\partial\mathcal{D}} \vec{A} \cdot d\vec{r}$ for $\vec{A} = \langle 4z, 5x, 3y\rangle$. (Note that: $\operatorname{curl}(\vec{A}) = \vec{F}$).



The plane can be parametrized as

$\vec{\mathbf{G}}\,(u,v) = \langle 0,0,5\rangle + u\,\langle 5,0,0\rangle + v\,\langle 0,5,0\rangle$ for $u \in [0,1]$ and $v \in [0,1]$.

Since $\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v = \langle 0,0,25\rangle$,

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \int_0^1\int_0^1 \langle 3,4,5\rangle \cdot \langle 0,0,25\rangle \;du\,dv$$

$$= \boxed{125}.$$

Alternatively, the following parametrization can be used:

$\vec{H}(u,v) = \langle 0,0,5\rangle + u\,\langle 5,0,0\rangle + v\,\langle 5,5,0\rangle,\;\; 0 \le v \le 1$ and $-v \le u \le 1-v$.

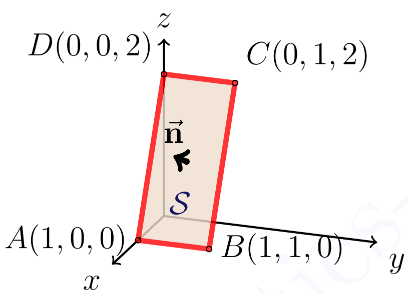
2. (A) Parameterize the following rectangular region using the formula: $\vec{\mathbf{G}}(u,v) = \overrightarrow{OA} + \overrightarrow{AB}\,u + \overrightarrow{AD}\,v$, for $0 \le u, v \le 1$.

The plane can be parameterized as

$\vec{\mathbf{G}}\,(u,v) = \langle 1,0,0\rangle + u\,\langle 0,1,0\rangle + v\,\langle -1,0,2\rangle$ for $u \in [0,1]$ and $v \in [0,1]$.

$\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v = \langle 2,0,1\rangle$

(B) Compute $\vec{\mathbf{G}}_u$, $\vec{\mathbf{G}}_v$ and $\vec{\mathbf{N}} = \vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v$ for Part (A). Is $\vec{\mathbf{N}}$ oriented in the same direction as $\vec{\mathbf{n}}$ or in the opposite direction?



$\vec{\mathbf{G}}_u = \langle 0,\ 1,\ 0\rangle$ and $\vec{\mathbf{G}}_v = \langle -1,\ 0,\ 2\rangle$.

$\vec{\mathbf{G}}_u \times \vec{\mathbf{G}}_v = \langle 2,0,1\rangle$

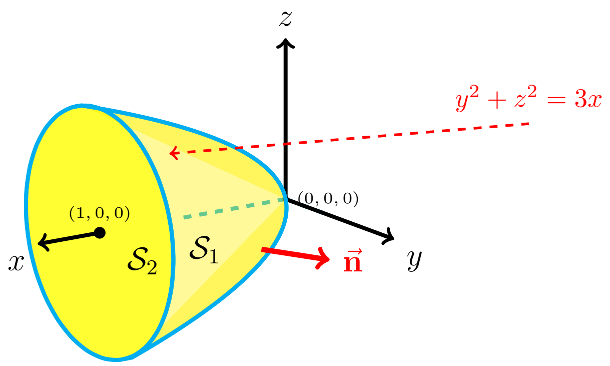
(C) Let $\vec{\mathbf{F}}(x,y,z) = \langle 3x, 4y, 5\rangle$. Parameterize $\vec{\mathbf{F}}$ on the surface and compute integral

$$\int_0^1\int_0^1 \vec{\mathbf{F}}(\vec{\mathbf{G}}(u,v))\cdot\vec{\mathbf{N}}\,du\,dv.$$

$\vec{\mathbf{F}}(\vec{\mathbf{G}}(u,v)) = \langle 3(1-1v), 4v, 5\rangle$

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \int_0^1\int_0^1 \langle 3(1-1v), 4u, 5\rangle \cdot \langle 2,0,1\rangle \;du\,dv = \boxed{8}.$$

3. Find the flux of $\vec{F}(x, y, z) = \langle z, x, y\rangle$ through the cap cut from the paraboloid $y^2 + z^2 = 3x$ by the plane $x = 1$, oriented as shown.



Solution 1: Solve directly. Parameterize the surface on $r \in [0, \sqrt{3}],\ \theta \in [0, 2\pi]$:

$$\vec{\mathbf{G}}(\theta, r) = \left\langle \frac{r^2}{3}, r\cos(\theta), r\sin(\theta) \right\rangle \qquad \vec{\mathbf{G}}_\theta \times \vec{\mathbf{G}}_r = \left\langle -r, \frac{2r^2}{3}\cos(\theta), \frac{2r^2}{3}\sin(\theta) \right\rangle$$

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\sqrt{3}} -r^2\sin(\theta) + \frac{2}{9}r^4\cos(\theta) + \frac{2}{9}r^4\sin(\theta)\cos(\theta)\, dr\, d\theta = 0$$

Solution 2: Solve using the Divergence Theorem. Note that $\operatorname{div}\left(\vec{F}\right) = 0$. Let $\mathcal{T}$ be the solid enclosed by $y^2 + z^2 = 3x$ **and** $x = 1$; note that $\partial\mathcal{T} = \mathcal{S}_1 + \mathcal{S}_2$, where $\mathcal{S}_1$ is the surface we are interested in.

Using the Divergence Theorem,

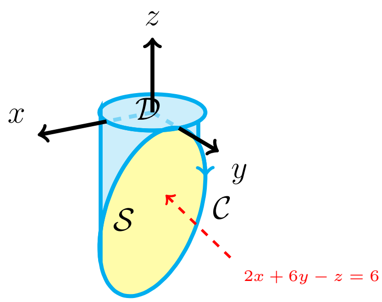
$$\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S} + \iint_{\mathcal{S}_2} \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{T}} \operatorname{div}\left(\vec{F}\right)\, dV = 0$$

Therefore, the flux through $\mathcal{S}_1$ is opposite the flux through $\mathcal{S}_2$. Parametrizing $\mathcal{S}_2$:

$$\vec{r}(r, \theta) = \langle 1, r\cos(\theta), r\sin(\theta)\rangle \qquad \vec{r}_r \times \vec{r}_\theta = \langle r, 0, 0\rangle \qquad r \in [0, \sqrt{3}],\ \theta \in [0, 2\pi]$$

$$\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S} = -\iint_{\mathcal{S}_2} \vec{F} \cdot d\vec{S} = -\int_0^{2\pi} \int_0^{\sqrt{3}} r^2\sin(\theta)\, dr\, d\theta = 0$$

4. Find the circulation of $\vec{F}(x,y,z) = \langle y^2, -y, 3z^2\rangle$ through the ellipse formed from $2x+6y-3z=6$ intersecting $x^2+y^2=1$, oriented counterclockwise as viewed from above.



Parametrize the surface of the intersection (the surface of the plane $z = \frac{2}{3}x + 2y - 2$ inside the ellipse) with an upwards orientation:

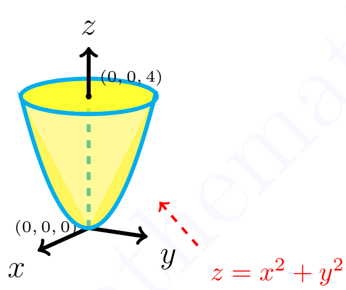
$$\vec{\mathbf{G}}\,(r,\theta) = \left\langle r\cos(\theta), r\sin(\theta), \frac{2}{3}r\cos(\theta) + 2r\sin(\theta) - 2\right\rangle \qquad \vec{\mathbf{G}}_r \times \vec{\mathbf{G}}_\theta = \left\langle \frac{-2r}{3}, -2r, r\right\rangle$$

Using Stokes' Theorem,

$$\int_{\partial\mathcal{S}} \vec{F}\cdot d\vec{r} = \iint_{\mathcal{S}} \text{curl}\left(\vec{F}\right)\cdot d\vec{S} = \iint_{\mathcal{S}} \langle 0,0,-2y\rangle\cdot d\vec{S}$$

$$= \int_0^{2\pi}\int_0^1 -2r^2\sin(\theta)\,dr\,d\theta = 0$$

5. Find the flux of $\vec{F}(x,y,z) = \langle x^2y, xy^2, 2xyz\rangle$ outward through the surface of solid bounded by the paraboloid $z = x^2+y^2$ and the plane $z=4$.



Note that the surface is closed. Using the Divergence Theorem,

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iiint_{\mathcal{T}} \text{div}\left(\vec{F}\right)\,dV = \iiint_{\mathcal{T}} 6xy\,dV$$

$$= \int_0^{2\pi}\int_0^2\int_{r^2}^4 6r^2\cos(\theta)\sin(\theta)\,r\,dz\,dr\,d\theta = 0$$